COMP6216

Gaussian in Lecture Slides

$$2x + y - z = 8$$
$$-3x - y + 2z = -11$$
$$-2x + y + 2z = -3$$
$$\Downarrow$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right) \begin{array}{ll} (1) & \cancel{\pm (3)} \\ (2) & +\tfrac{3}{2}(1) \\ (3) & +(1) \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} & 1 \\ 0 & 2 & 1 & 5 \end{array}\right) \begin{array}{ll} (1) & \\ (2') & \times 2 \\ (3') & - 4(2') \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

$$-z = 1 \Rightarrow z = -1$$
$$y + z = 2 \qquad y = 3$$
$$2x + y - z = 8 \qquad x = 2$$

Problem at End

Gaussian

$$5x + y = 1$$
$$2x + 3y = 1$$

$$\left(\begin{array}{cc|c} 5 & 1 & 1 \\ 2 & 3 & 1 \end{array}\right) \begin{array}{ll} (1) & \\ (2) & - \tfrac{2}{5}(1) \end{array}$$

$$\left(\begin{array}{cc|c} 5 & 1 & 1 \\ 0 & \tfrac{13}{5} & \tfrac{3}{5} \end{array}\right) \begin{array}{ll} (1) & \\ (2') & \times 5 \end{array}$$

$\cancel{5x + y = 1}$ $$\left(\begin{array}{cc|c} 5 & 1 & 1 \\ 0 & 13 & 3 \end{array}\right)$$

$$13y = 3$$
$$y = \tfrac{3}{13}$$

$$5x + \tfrac{3}{13} = 1$$
$$5x = \tfrac{10}{13}$$
$$x = \tfrac{2}{13}$$

Jacobi

$$5x + y = 1 \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$2x + 3y = 1$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \quad R = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_{n+1} = D^{-1}(b - Rx_n) \Rightarrow c = D^{-1}b$$

$$D^{-1} = \frac{1}{15}\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \qquad T = D^{-1}R$$

$$= \begin{pmatrix} 1/5 & 0 \\ 0 & 1/3 \end{pmatrix}$$

$$c = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 1/3 \end{pmatrix}$$

$$T = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/5 \\ 2/3 & 0 \end{pmatrix}$$

$$x_{n+1} = c - Tx_n$$

$$x_1 = \begin{pmatrix} 1/5 \\ 1/3 \end{pmatrix} - \begin{pmatrix} 0 & 1/5 \\ 2/3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/5 \\ 1/3 \end{pmatrix} - \begin{pmatrix} 1/5 \\ 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1/3 \end{pmatrix} \checkmark$$

$$x_2 = \begin{pmatrix} 1/5 \\ 1/3 \end{pmatrix} - \begin{pmatrix} 0 & 1/5 \\ 2/3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 0.267 \\ 0.333 \end{pmatrix} \quad (3.s.f.)$$

(by calculator)

$x_3 = (0.133, 0.155)^T$

$x_4 = (0.169, 0.244)^T$

$x_5 = (0.151, 0.220)^T$

$x_6 = (0.156, 0.233)^T$

$x_7 = (0.153, 0.229)^T$

$x_8 = (0.154, 0.231)^T$

$x_9 = (0.154, 0.231)^T$ (same to 3.s.f.)

Gauss Solutions: $x = 0.154$

$y = 0.231$ (3.s.f.)

$\therefore \lim_{n \to \infty} x, y_{\text{Jacobi}} = x, y_{\text{Gauss}}$.

(to show independence of starting conditions, choose a different x_0).

Problem Sheet 1

Q1 Gaussian Elimination

	Check	Check 2nd Time
$6x + y + z = 11$	✓	✓
$2x + 3y - z = 5$	✓	✓
$x - y - 2z = -7$	×	✓

$$\left(\begin{array}{ccc|c} 6 & 1 & 1 & 11 \\ 2 & 3 & -1 & 5 \\ 1 & -1 & -2 & -7 \end{array}\right) \begin{array}{l} (1) \\ (2) - \tfrac{1}{3}(1) \\ (3) - \tfrac{1}{2}(2) \end{array}$$

$$\left(\begin{array}{ccc|c} 6 & 1 & 1 & 11 \\ 0 & 8/3 & -\cancel{2}4/3 & 4/3 \\ 0 & -5/2 & -\cancel{11}3/2 & -19/2 \end{array}\right) \begin{array}{ll} (1) & \\ (2') & \times 3 \\ (3') & \times 2 \end{array}$$

$$\left(\begin{array}{ccc|c} 6 & 1 & 1 & 11 \\ 0 & 8 & -\cancel{2}4 & 4 \\ 0 & -5 & -\cancel{8}3 & -19 \end{array}\right) \begin{array}{ll} (1) & ✓ \\ (2'') & ✓ \\ (3'') & + \tfrac{5}{8}(2'') \end{array}$$

$$\left(\begin{array}{ccc|c} 6 & 1 & 1 & 11 \\ 0 & 8 & -\cancel{2}4 & 4 \\ 0 & 0 & -\cancel{25}11/2 & -33/2 \end{array}\right) ✓$$

~~$-\tfrac{25}{4}z = -\tfrac{33}{2}$~~

~~$z = 825/8$~~

$-\cancel{15}11/2\, z = -33/2$

$-\cancel{15}11 z = -33$

$z = 33/11$

$= 3$

$8y - 4(3) = 4$

$8y = 16$

$y = 2$

$6x + 2 + 3 = 11$

$6x = 6$

$x = 1$

Very error prone!

Q2 Jacobi Iteration

$$5x + y = 1 \qquad A = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$2x + 3y = 1$$

$$x_{n+1} = D^{-1}(b - Rx) = D^{-1}b - D^{-1}Rx$$

For Jacobi valid:

$$|a_{ii}| > \sum_j |a_{ij}| \quad \text{(diagonal)}.$$

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \qquad D^{-1} = \frac{1}{15}\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}.$$

This is the same problem as at end of linear systems of equations lecture.

Q3 Newton 1 $\quad f(x) = x^3 \quad f'(x) = 3x^2$

$$x^3 = 0 \qquad x_0 = 1 \qquad \epsilon = 10^{-5}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{x_k^3}{3x_k^2}$$

$$x_1 = 1 - \frac{1^3}{3(1)^2}$$

$$= 2/3 = \cancel{1.333}\ 0.666$$

$$x_2 = 4/3 + \frac{(4/3)^3}{3(4/3)^2}$$

$$= 4/3 + \frac{64/27}{16/3}$$

$$= \frac{4}{3} + \frac{64}{27} \times \frac{3}{16}$$

$$= \frac{4}{3} + \frac{1024}{81}$$

$$= \frac{1132}{81} = 13.98 \text{ (4.s.f.)}$$

$$\frac{m^k}{1-m}|x_1 - x_0| \le \epsilon$$

$$m^k|x_1 - x_0| \le \epsilon(1-m)$$

$$\ln(m^k|x_1 - x_0|) \le \ln(\epsilon(1-m))$$

(not part of problem sheet)

$$\ln(m^k) + \ln(|x_1 - x_0|) \le \ln(\epsilon(1-m))$$

$$\ln(m^k) \le \ln(\epsilon(1-m)) - \ln(|x_1 - x_0|)$$

$$k\ln(m) \le \ln(\epsilon(1-m)) - \ln(|x_1 - x_0|)$$

$$k \le \frac{\ln(\epsilon(1-m)) - \ln(|x_1 - x_0|)}{\ln(m)}$$

(by calc)

$x_3 = 18.63$
$x_4 = 24.84$
$x_5 = 33.13$
$x_6 = 44.17$
$x_7 = 58.89$ (used wrong eqn)
$x_8 = 78.52$
$x_9 = 104.7$
$x_{10} = 139.6$
$x_{11} = 186.1$
$x_{12} = 248.2$

$(\epsilon = 10^{-5})$

$x_3 = 0.44444$
$x_4 = 0.29627$
$x_5 = 0.19753$
$x_{10} = 0.02601$
$x_{15} = 0.00034255$
$x_{20} = 0.000045109$
$x_{24} = 0.0000089105$

(Something's wrong!)

$$x_{(k+1)} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^{\cancel{3}}}{3x_k^{\cancel{2}}} = \frac{2x_k}{3}$$

$$x_{k+1} = \frac{2}{3} x_k$$

(rearrange to get x_k and x_{k-1})

$$x_k = \frac{2}{3} x_{k-1} = \frac{2}{3}\left(\frac{2}{3} x_{k-2}\right)$$

$$= \left(\frac{2}{3}\right)^k x_0$$

(solve for $k \leq \epsilon$)

$$\left(\frac{2}{3}\right)^k \leq 10^{-5}$$

This is negative, so ÷ will flip the sign round.

$$k \ln(2/3) \leq \ln(10^{-5})$$

$$k \leq \frac{\ln(10^{-5})}{\ln(2/3)}$$

$$k \leq \frac{-5 \ln 10}{\ln(2/3)}$$

(could also do this) $\frac{5 \ln 10}{\ln(3/2)}$ as $\ln(3/2) = -\ln(2/3)$.

$$k \geq 28.394$$

$k = 29$ iterations

Q5 Multidimensional Newton (Looks painful!)

$Ax = b \qquad x, b \in \mathbb{R}^d$. (Linear system of equations)

$$e_{k+1} = x_{k+1} - r$$

Bisection — doesn't work

FPI — $\vec{x}_k = \vec{g}(\vec{x}_{k-1})$

Newton — see below

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Generally, to extract k, rearrange for x_k:

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}.$$

$$= \left(k_{k-2} - \frac{f(x_{k-2})}{f'(x_{k-2})} \right) - \frac{f\left(x_{k-2} - \frac{f(x_{k-3})}{f'(x_{k-3})}\right)}{f'\left(x_{k-2} - \frac{f(x_{k-3})}{f'(x_{k-3})}\right)}$$

(Horrible!)

$$x_{k+1} = x_k - (DF(x_k))^{-1} F(x_k).$$

Not completed!

Q4 Using FPI

Recall FPI requires $|g'| \leq 1$ to converge. For

$$x^2 - 3x = -2 \Rightarrow f(x) = x^2 - 3x + 2 = 0$$

$$\Rightarrow g(x) = f(x) + x = x \qquad (x-1)(x-2) = 0$$

$$x = 1, 2$$

$$= x^2 - 2x + 2$$

$x_0 = 1$ is a solution.

$$g'(x) = 2x - 2$$

$$g'(x_0) = 2(1) - 2$$

$$= 0.$$

$x_0 = 2$ is a solution.

$$g'(x_0) = 2(2) - 2$$

$$= 2.$$

Will not converge.

Lecture notes go a different way!

$$x^2 - 3x + 2 = 0$$

$$3x = x^2 + 2$$

$$x = \tfrac{1}{3}(x^2 + 2)$$

$$g(x) = \tfrac{1}{3}(x^2 + 2)$$

Then feeds into their $g'(x)$.

My solution also works fine!